

Department of Mathematics
Pattamundai College, Pattamundai

3rd Semester

Group Theory (Algebra-II)

Core – 6

Sec–A

(Unit–1)

1. Write the definition of group.
2. Show that $(\mathbb{Z}, +)$ is an abelian group.
3. Show that (\mathbb{N}, \cdot) is a semi group.
4. Show that the inverse element of a group is unique.
5. Show that the identity element in a group is unique.
6. Write the difference between the order of the element and order of the group.
7. Give an example of a group which is not an abelian.
8. Find the order of the element 2 in \mathbb{Z}_6 .
9. Show that every cyclic group is an abelian.
10. The number of element of order 2 in D_{11} .
11. Find the number of generator of the group \mathbb{Z}_{12} .
12. Find the order of the element $\sigma = (123)$.
13. If G is a group and H is a sub group of G and H is called a sub group of $G \Leftrightarrow ab^{-1} \in H$, for all $a, b \in H$.
14. In $GL_2(\mathbb{R})$, find the order of the element $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$
15. $SL_2(\mathbb{R}) \subseteq GL_2(\mathbb{R})$ (T/F)

(Unit–2)

16. Show that every sub group of cyclic group is cyclic.
17. Give an example of a group which is an abelian but not cyclic.
18. Show that in a cyclic group the generator is not unique.
19. Show that $(\mathbb{Q}, +)$ is a sub group of $(\mathbb{R}, +)$ but it is not cyclic.
20. Write the definition of centralizer of a group.
21. Write the definition of Normalizer of a group.
22. Write the definition of centre of a group.
23. Find the order of (A_n) .
24. Write the statement of Lagrange theorem.

P.T.O.

25. Write the statement of Fermats little theorem.
26. Show that every abelian groups are normal.
27. Show that a group has index is 2, then the group must be normal.
28. If $F = f_1 f_2 f_3 \dots f_k$ where $f_1, f_2, f_3, \dots, f_k$ are distinct permutation then $O(F) = ?$
29. Find the number of the element of order 2 in S_5 .

(Unit-3)

30. Define Kernal of homomorphism.
31. Let ϕ be homomorphism from a group G to the group \bar{G} . Let $g \in G$ then $\phi(g^n) = (\phi(g))^n$.
32. If H is a cyclic then $\phi(H)$ is cyclic, prove it.
33. Show that if H is an abelian group then $\phi(H)$ is an abelian group.
34. If ϕ is onto and $\ker \phi = \{e\}$ then show that ϕ is an isomorphism from G to \bar{G} .
35. If G is infinite cyclic group then show that $G \cong \mathbb{Z}$.
36. If $|G| = n$ is a cyclic group then $G \cong \mathbb{Z}_n$.
37. Show that $GL_2(\mathbb{Z}_2) \cong S_3$
38. Show that $\mathbb{Z}_6 \cong \mathbb{Z}_2 \times \mathbb{Z}_3$
39. Show that $\mathbb{Z}_4 \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$

(Unit-4)

40. Write only the statement of first isomorphism theorem.
41. Write only the statement of second isomorphism theorem.
42. Write only the statement of 3rd isomorphism theorem.
43. Determine all homomorphisms from \mathbb{Z}_{20} onto \mathbb{Z}_{10} .
44. Find all cyclic subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_4$
45. Find all cyclic subgroups of $\mathbb{Z}_3 \times \mathbb{Z}_6$
46. Let G be a group all let $a \in G$ be an element of order 30. List the power of a that have order 2 and order 3.
47. Find the number of sub group of \mathbb{Z}_{12} .
48. Write the statement of cayleys theorem.
49. Show that the inverse of a group isomorphism is a group isomorphism.
50. show that the composition of two isomorphisms, is a group isomorphism.

[3]

Sec-B
(Unit-1)

1. Show that $(\mathbb{R}, +)$ is an abelian group.
2. Show that (\mathbb{Z}_3, \cdot) is an abelian group
3. $G = U(8) = \{1, 3, 5, 7\}$. Show that $U(8)$ is a group w.r.t multiplication modulo 8?
4. Find possible order of the elements in D_{10} .
5. Find possible order of the elements in Z_{10} .
6. How many elements of order finite in \mathbb{Q}^* ?
7. Let G be a group with each element of order 2 then show that G is an abelian group.
8. Let $G = \{e, a, b, ab \mid a^2=e, b^2=e, ab = ba\}$. Then show that G is a group.
9. Show that $O(a) = O(xax^{-1}) = O(x^{-1}ax)$, for all $x, a \in G$.
10. Let G be a group with $(ab)^2 = a^2b^2$, If and only if G is an abelian.
11. Find the element and possible order of element in $Z_2 \times Z_3$.
12. Show that the centre of a group is a subgroup of G .
13. Show that G is an abelian then $z(G)=G$.
14. Show that $Z(G)$ is always an abelian group.
15. Let H and K be two subgroup of G then show that $O(HK) = \frac{O(H) \cdot O(K)}{O(H \cap K)}$

(Unit-2)

16. Show that $G = \{1, w, w^2, \mid w^3=1\}$ is a cyclic group.
17. Show that $G = \{1, -1, i, -i \mid i^2 = -1\}$ is a cyclic group.
18. Show that $Z_2 \times Z_3$ is a cyclic group.
19. Show that $U(100)$ is not cyclic but abelian group.
20. Show that S_3 is not a cyclic group.
21. Show that $O(G) = p^2$, where P is any prime number than G is an abelian but not cyclic group.
22. Show that all the sub group of a cyclic group is cyclic.
23. Let $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 3 & 4 & 1 & 6 & 5 & 8 & 9 & 7 \end{pmatrix} \in S_n, n \geq 9$ then find $o(f) = ?$

P.T.O.

24. If, $f \in S_n$ is an even permutation then show that f^{+1} is an even permutation.
25. Find number of elements of order 2 in S_4 .
26. If $\beta = (1357986)(2410) \in S_{10}$, find smallest positive integer m such that $\beta^m = \beta^{-5}$.
27. Show that A_4 has no sub group of order 6.
28. If $O(G) = 10$ and G is an abelian group then how many element of order 10 in G .
29. How many elements of order 2 in $S_4 \times Z_2$?

(Unit-3)

30. If G is an abelian group then show that left coset of H in G is equal to right coset of H in G .
31. Let G be a cyclic group of order 6 with generated by 'a' and H be a subgroup of G generated by a^2 , Find the right coset of H in G .
32. Let $G = Z$, $H = 3Z$, find coset of H in G .
33. Show that the converse of Lagrange's theorem is not true.
34. If $G = Z_2 \times Z_2$. How many normal sub group in G ?
35. Show that G is any cyclic group, then all the sub group are normal.
36. Define the index of subgroup. Show that the sub group of index 2 is always normal.
37. Prove that let G be an finite abelian group and let 'p' be a prime that divides the order of G , then G has an element of order P .
38. Let G be any group and let $Z(G)$ be the centre of G if $\frac{G}{Z(G)}$ is cyclic then G is an abelian.
39. Let G be a group and let $a \in G$ prove that $\langle a^{-1} \rangle = \langle a \rangle$

(Unit-4)

40. Let $G = GL(2, R)$ and R^* be the group of non zero real numbers under multiplication. Then $\phi(A) = \det A$ is a homomorphism from G to R^* . Find the Kernel of ϕ .
41. Let G be a group and let a be any element of G . Define $\phi : Z \rightarrow G$ by $\phi(a) = a^n$. for all $n \in Z$ show that ϕ is a homomorphism.
42. Find the number of homomorphism from $f : Z_2 \times Z_4 \rightarrow Z_2 \times Z_4$
43. Show that the Kernel of a homomorphism is a normal subgroup of G .
44. If $f : Z_8 \rightarrow Z_4$ and $f(x) = 2x$ is not a onto homomorphism.
45. Find all the onto homomorphism $f : Z_{10} \rightarrow Z_5$
46. Show that there are infinite number of homomorphism from Z to Z
47. Let $\phi : G \rightarrow \overline{G}$ be a homomorphism. If ϕ is onto and $\text{ker } \phi = \{e\}$, then ϕ is an isomorphism from G to \overline{G} .

48. Let $\phi : G \rightarrow \bar{G}$ be a homomorphism. If \bar{K} is normal sub group of \bar{G} then $\phi^{-1}(\bar{K}) = \{k \in G | \phi(k) \in \bar{K}\}$ is a normal sub group of G .
49. Let $\phi : G \rightarrow \bar{G}$ be a homomorphism. If H is normal in G then $\phi(H)$ is normal in $\phi(G)$.
50. Let $\phi : G \rightarrow \bar{G}$ be a homomorphism. If H is cyclic then $\phi(H)$ is cyclic.

Sec-C
(Unit-1)

1. Show that D_4 is a group under composition.
2. Show that $SL_n(\mathbb{F}) = \{A = (a_{ij})_{n \times n} / \det(A) = 1, a_{ij} \in \mathbb{F}\}$ is a group matrix multiplication.
3. Show that quaternion (Q_4) is a group w.r.t multiplication, where
 $Q_4 = \{\pm 1, \pm i, \pm j, \pm k / i^2 = j^2 = k^2 = -1, i.j = -j.i, j.k = -k.j, k.i = -i.k\}$
4. Let $a \in G$ and $O(a) = 5$ such that $b^2 = a b a^{-1}$, $e \neq b \in G$, then find $O(b)$?
5. An abstract algebra teacher intended to give a typist a list of nine integers that form a group under multiplication modulo 91. Instead, one of the nine integers was inadvertently left out so that the list appeared as 1, 9, 16, 22, 53, 74, 79, 81. Which integer was left out?

(Unit-2)

6. Prove that $H = \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \mid n \in \mathbb{Z} \right\}$ is a cyclic sub group of $GL(2, \mathbb{R})$.
7. Let a and b belong to a group. If $|a| = 12$, $|b| = 22$ and $\langle a \rangle \cap \langle b \rangle = \{e\}$. Prove that $a^6 = b^{11}$.
8. Prove that S_n is non-Abelian for all $n \geq 3$.
9. Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 3 & 5 & 4 & 7 & 6 & 8 \end{pmatrix}$ and $\beta = \begin{pmatrix} 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 3 & 8 & 7 & 6 & 5 & 2 & 4 \end{pmatrix}$ Compute each of the following
 a) α^{-1} (b) β^α (c) $\alpha\beta$
10. Let α and β belong to S_n . Prove that $\alpha^{-1} \beta^{-1} \alpha \beta$ is an even permutation.

(Unit-3)

11. Let $G = Z_4 \times Z_2$ and $H = \langle (2, 1) \rangle$ then $\frac{G}{H}$ is isomorphic to Z_4 .
12. What is the order of any nonidentity element of $Z_3 \oplus Z_3 \oplus Z_3$.
13. Let G be a group and H a normal subgroup of G . The set $\frac{G}{H} = \{aH \mid a \in G\}$ is a group under the operation $(aH)(bH) = abH$. Prove that $\frac{G}{H}$ is a group.

P.T.O.

14. In Z , let $H = \langle 5 \rangle$ and $k = \langle 7 \rangle$. Prove that $Z \cong HK$. Does $Z = H \times K$?
15. Prove that a factor group of an Abelian group is Abelian.

(Unit-4)

11. Let G be a group, $a \in G$, $T_a: G \rightarrow G$ defined by $T_a(x) = axa^{-1}$, then show that
- 1) T_a is homomorphism
 - 2) T_a is one-one
 - 3) T_a is onto.
17. Let $f: G \rightarrow \bar{G}$ is a homomorphism and $O(G) = 20$, $O(\bar{G}) = 25$. Find possible order of $\ker f$?
- a) 1 (b) 2 (c) 3 (d) 4
18. Let G be a group of permutations. For each σ in G , define

$$\text{sgn}(\sigma) = \begin{cases} +1 & \text{if } \sigma \text{ is an even permutation} \\ -1 & \text{if } \sigma \text{ is an odd permutation} \end{cases}$$

Prove that sgn is a homomorphism from G to the multiplicative group $\{+1, -1\}$. What is the Kernel?

19. Suppose that ϕ is a homomorphism from $U(40)$ to $U(40)$ and $\ker \phi = \{1, 9, 17, 33\}$. If $\phi(11) = 11$, find all elements of $U(40)$ that map to 11.

20. If M and N are normal subgroups of G and $N \leq M$. Prove that $\frac{\left(\frac{G}{N}\right)}{\left(\frac{M}{N}\right)} \cong \frac{G}{M}$

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